Partial Derivatives - Completely Done

Partial derivatives are an important mathematical tool in a number of physics disciplines, most notably field theories (e.g. electricity & magnetism and general relativity) and in thermodynamics.

However, working with partial derivatives are always a bit tricky and teaching students about them is usually fraught with difficulties.  So it was to my pleasant surprise that I found a really nice discussion of how to derive the various 'classic' rules cleanly presented in *Classical and Statistical Thermodynamics* by Ashley H. Carter.

My presentation here is strongly influenced and closely follows her presentation in Appendix A, although I've added on a bit in the theoretical flow and I've also provided explicit examples in terms of the standard paraboloid found in freshman calculus.

Assume a function of 3 variables can be expressed as $$f(x,y,z) = 0$$.  This equation can be viewed as a constrain equation linking the values of the variables such that two variables are independent.  That means that we can (at least locally) solve for:

* $$x(y,z) = 0$$
* $$y(x,z) = 0$$
* $$z(x,y) = 0$$

Focus on the first and second forms (other pairings will follow a simple relabeling of the variables).  The corresponding differentials are

\[ dx = \left( \frac{\partial x}{\partial y} \right)\_z dy + \left( \frac{\partial x}{\partial z} \right)\_y dz \; \]

and

\[ dy = \left( \frac{\partial y}{\partial x} \right)\_z dx + \left( \frac{\partial y}{\partial z} \right)\_x dz \; .\]

Now substitute the expansion of $$dy$$ into the expansion of $$dx$$

\[ dx = \left(\frac{\partial x}{\partial y}\right)\_z \left[ \left(\frac{\partial y}{\partial x}\right)\_z dx + \left(\frac{\partial y}{\partial z}\right)\_x dz \right] + \left(\frac{\partial x}{\partial z}\right)\_y dz \; ,\]

which simplifies to

\[ dx = \left(\frac{\partial x}{\partial y}\right)\_z \left(\frac{\partial y}{\partial x}\right)\_z dx + \left[ \left(\frac{\partial x}{\partial y}\right)\_z \left(\frac{\partial y}{\partial z}\right)\_x + \left(\frac{\partial x}{\partial z}\right)\_y \right] dz \; .\]

Putting it all together gives

\[ \left[ 1 - \left(\frac{\partial x}{\partial y}\right)\_z \left(\frac{\partial y}{\partial x}\right)\_z \right] dx - \left[ \left(\frac{\partial x}{\partial y}\right)\_z \left(\frac{\partial y}{\partial z}\right)\_x + \left(\frac{\partial x}{\partial z}\right)\_y \right] dz = 0 \; .\]

Since $$dx$$ and $$dz$$ are independent, each differential can be set to zero independently, giving one of the classic identities.

First set $$dz = 0$$ to get

\[ \left(\frac{\partial x}{\partial y}\right)\_z = 1/ \left(\frac{\partial y}{\partial x}\right)\_z \; , \]

which is called the reciprocal rule.

Next, setting $$dx = 0$$ yields

\[ \left(\frac{\partial x}{\partial y}\right)\_z \left(\frac{\partial y}{\partial z}\right)\_x = \; - \; \left(\frac{\partial x}{\partial z}\right)\_y \; , \]

which is called the fraction rule.

The manipulations are complete when using the reciprocal rule in the fraction rule and simplify to get

\[ \left(\frac{\partial x}{\partial y}\right)\_z \left(\frac{\partial y}{\partial z}\right)\_x \left(\frac{\partial z}{\partial x}\right)\_y = -1 \; , \]

which is called the cyclic rule.

Let’s take a look at these relationships in action. Consider the implicit definition of the paraboloid

\[ x^2 + y^2 - z = 0 \; . \]

As mentioned earlier, this equation can be considered as a constraint equation that selects out a value for any one of the three variables given the other two. In other words, we can imagine a look up table where we select a value of $$x$$ and $$y$$, we rummage through the table to find a row with both values and then we scan to the right to find the allowed value of $$z$$ that makes it satisfy the implicit equation.

How do you construct this table; not at a finite set of points but functionally so that it works at any point? It is natural and easy to determine $$z$$ given $$x$$ and $$y$$ by simply rewriting the implicit equation as

\[ z(x,y) = x^2 + y^2 \; .\]

However, it isn’t as easy to express $$x$$ or $$y$$ as functions of the remaining two variables because of the two possible signs that result from taking the square root. We need to have four functional relationships

\[ x\_p(y,z) = \sqrt{ z \; - \; y^2 } \; ,\]

\[ x\_n(y,z) = \; - \; \sqrt{z \; - \; y^2} \; , \]

\[ y\_p(x,z) = \sqrt{z \; - \; x^2} \; , \]

and

\[ y\_n(x,z) = \; - \; \sqrt{z \; - \; x^2} \; , \]

depending on the particular combination of whether $$x$$ is positive or negative and whether $$y$$ is also positive or negative.  In the language of differential geometry, we have a 5 charts in our atlas.

We are now in position to try the various relations derived above. For example, let's examine the reciprocal relation in the first quadrant of the $$x$$-$$y$$ plane.  We need to use $$x\_p$$ as our local chart.

\[ \left(\frac{\partial x\_p}{\partial z}\right)\_y = \frac{1}{2} \frac{1}{\sqrt{z - y^2}} \; \]

or once we recognize the denominator as $$x\_p$$

\[\left(\frac{\partial x\_p}{\partial z}\right)\_y = \frac{1}{2 x\_p} \; . \]

The 'reciprocal' partial derivative is

\[ \left( \frac{\partial z}{\partial x} \right)\_y = 2 x = 2 x\_p \; ,\]

where there is no need for the $$z$$-chart to distinguish between positive and negative values of $$x$$.  As expected the derivatives are reciprocals of each other.

Next, let's test the fraction rule.  For fun, this time let's test it in the 2nd quadrant in the $$x$$-$$y$$ plane ($$x < 0$$ and $$y > 0$$).  Calculating the partial derivatives on the left-hand side yields

\[ \left(\frac{\partial x\_n }{\partial y\_p } \right)\_z = \frac{y\_p}{\sqrt{z - y\_p^2}} = \; - \; \frac{y\_p}{x\_n} \; \]

and

\[ \left(\frac{\partial y\_p }{\partial z} \right)\_{x\_n} = \frac{1}{2\sqrt{z-x\_n^2}} = \frac{1}{2 y\_p} \; . \]

It is a simple matter to verify that

\[ \left(\frac{\partial x\_n }{\partial y\_p} \right)\_z \left(\frac{\partial y\_p }{\partial z} \right)\_{x\_n} = -\frac{1}{2 x\_n} \; \]

is identical to

\[ - \left(\frac{\partial x\_n }{\partial z} \right)\_{y\_p} = \frac{1}{2 \sqrt{z - y\_p^2} } = -\frac{1}{2 x\_n} \; .\]

Finally, for the cyclic rule, let's go into the 4th quadrant in the $$x$$-$$y$$ plane ($$x>0$$ and $$y<0$$).  Taking each partial derivative in turns yields

\[ \left(\frac{\partial x\_p }{\partial y\_n} \right)\_{z} = -\frac{y\_n}{\sqrt{z-y\_n^2}} = -\frac{y\_n}{x\_p} \; ,\]

\[ \left(\frac{\partial y\_n }{\partial z} \right)\_{x\_p} = -\frac{1}{2 \sqrt{z-x\_p^2} } = \frac{1}{2 y\_n} \; ,\]

and

\[ \left(\frac{\partial z }{\partial x\_p} \right)\_{y\_n} = 2 x\_p \; .\]

Multiplying these terms in order gives

\[ -\frac{y\_n}{x\_p} \frac{1}{2 y\_n} 2 x\_p = -1 \; .\]

Nice, neat, and more than partially done.